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Matrix biorthogonal polynomials on the unit circle and non-Abelian Ablowitz–Ladik hierarchy

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Abstract

Adler and van Moerbeke (2001 *Commun. Pure Appl. Math.* **54** 153–205) described a reduction of the 2D-Toda hierarchy called the Toeplitz lattice. This hierarchy turns out to be equivalent to the one originally described by Ablowitz and Ladik (1975 *J. Math. Phys.* **16** 598–603) using semidiscrete zero-curvature equations. In this paper, we obtain the original semidiscrete zero-curvature equations starting directly from the Toeplitz lattice and we generalize these computations to the matrix case. This generalization leads us to the semidiscrete zero-curvature equations for the non-Abelian (or multicomponent) version of the Ablowitz–Ladik equations (Gerdzhikov and Ivanov 1982 *Theor. Math. Phys.* **52** 676–85). In this way, we extend the link between biorthogonal polynomials on the unit circle and the Ablowitz–Ladik hierarchy to the matrix case.

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1. Introduction

Ablowitz–Ladik hierarchy was introduced in 1975 [1] as a spatial discretization of the Ablowitz–Kaup–Newell–Segur (AKNS) hierarchy. As described by Suris in [2] Ablowitz and Ladik replaced the celebrated Zakharov–Shabat spectral problem

$$\begin{cases} \partial_x \Psi = L\Psi \\ \partial_\tau \Psi = M\Psi \end{cases}$$

with a discretized version of it; namely

$$\begin{cases} \Psi_{k+1} = L_k \Psi_k \\ \partial_\tau \Psi_k = M_k \Psi_k. \end{cases}$$

Here Ψ and Ψ_k are two-component vectors while L, M, L_k and M_k are 2×2 matrices. In particular

$$L := \begin{pmatrix} z & x \\ y & -z \end{pmatrix}$$

and

$$L_k := \begin{pmatrix} z & x_k \\ y_k & z^{-1} \end{pmatrix}.$$

We can consider, as usual, the periodic case ($k \in \mathbb{Z}_n$), the infinite case ($k \in \mathbb{Z}$) or the semi-infinite case ($k \in \mathbb{N}$). The standard zero-curvature equations for the AKNS hierarchy are replaced with the semidiscrete zero-curvature equations

$$\partial_\tau L_k = M_{k+1} L_k - L_k M_k.$$

As an example, one of the most important equations of this hierarchy is the discrete complexified version of the nonlinear Schrödinger equations

$$\begin{cases} \partial_\tau x_k = x_{k+1} - 2x_k + x_{k-1} - x_k y_k (x_{k+1} + x_{k-1}) \\ \partial_\tau y_k = -y_{k+1} + 2y_k - y_{k-1} + x_k y_k (y_{k+1} + y_{k-1}). \end{cases}$$

Quite recently different authors (see [13, 15]) worked on the link between biorthogonal polynomials on the unit circle and the semi-infinite Ablowitz–Ladik hierarchy. This is an analogue of the celebrated link between the Toda hierarchy and orthogonal polynomials on the real line. Many articles have been written about orthogonal polynomials and the Toda hierarchy and it is almost impossible to recall all of them. Let us just mention the seminal paper of Moser [8] and the monograph [9]; there interested readers can find many references. The more recent articles [11–13] should also be cited. In these articles the connection between the theory of orthogonal polynomials and Sato's theory of infinite Grassmannians appeared for the first time. In particular, in [13] the case of Toda and Ablowitz–Ladik hierarchies is treated in a similar way to reductions of 2D-Toda. In this way a clear and unified explanation of the role played by orthogonal polynomials in the theory of integrable equations is given. As noted by the authors their approach is quite different from the original one; actually in their paper [13] they always refer to the Toeplitz lattice and the coincidence with the Ablowitz–Ladik hierarchy is only stated in the introduction. Nevertheless, as explained in section 2 of this paper, it is very easy to deduce semidiscrete zero-curvature equations starting from Adler–van Moerbeke's equations. With this approach in mind we addressed a question arising from the following facts.

- Time evolution for orthogonal polynomials on the real line leads to the Toda hierarchy.
- Time evolution for biorthogonal polynomials on the unit circle leads to the Ablowitz–Ladik hierarchy.
- Time evolution for matrix orthogonal polynomials on the real line leads to the non-Abelian Toda hierarchy (see for instance [17] and references therein; our approach is close to [20]).

What about time evolution for matrix biorthogonal polynomials on the unit circle?

In other words, our goal was to replace the question mark in the table below with the corresponding hierarchy.

	Orthogonal polynomials on $\mathbb{R}$	Biorthogonal polynomials on S^1
Scalar case	Toda	Ablowitz–Ladik
Matrix case	Non-Abelian Toda	?

In this paper, we prove that the relevant hierarchy is the non-Abelian version of the Ablowitz–Ladik hierarchy. This hierarchy has been studied by different authors since 1983 (see [3–5]) but, to the best of our knowledge, the connection with matrix biorthogonal polynomials on the unit circle has never been established before. We also remark that this hierarchy is usually called the matrix, vector or multicomponent Ablowitz–Ladik. Here we prefer to call it non-Abelian to stress the analogy with Toda. In our setting, instead of having one Lax operator L_k of size 2×2, we have two Lax operators $\mathcal{L}_k^l$ and $\mathcal{L}_k^r$ of size $2n \times 2n$. These operators depend on matrices x_k^l, x_k^r, y_k^l and y_k^r of size $n \times n$. Semidiscrete zero curvature equations are given by

$$\partial_\tau \mathcal{L}_k^l = \mathcal{M}_{k+1}^l \mathcal{L}_k^l - \mathcal{L}_n^l \mathcal{M}_k^l$$

$$\partial_\tau \mathcal{L}_k^r = \mathcal{L}_k^r \mathcal{M}_{k+1}^r - \mathcal{M}_k^r \mathcal{L}_k^r,$$

where $\mathcal{M}_k^l$ and $\mathcal{M}_k^r$ are also block matrices. For instance we have, in this hierarchy, two versions of the non-Abelian complexified discrete nonlinear Schrödinger:

$$\begin{cases} \partial_\tau x_k^l = x_{k+1}^l - 2x_k^l + x_{k-1}^l - x_{k+1}^l y_k^r x_k^l - x_k^l y_k^r x_{k-1}^l \\ \partial_\tau y_k^r = -y_{k+1}^r + 2y_k^r - y_{k-1}^r + y_{k+1}^r x_k^l y_k^r + y_k^r x_k^l y_{k-1}^r \end{cases}$$

$$\begin{cases} \partial_\tau x_k^r = x_{k+1}^r - 2x_k^r + x_{k-1}^r - x_{k-1}^r y_k^l x_k^r - x_k^r y_k^l x_{k+1}^r \\ \partial_\tau y_k^l = -y_{k+1}^l + 2y_k^l - y_{k-1}^l + y_{k-1}^l x_k^r y_k^l + y_k^l x_k^r y_{k+1}^l \end{cases}$$

(see [6] for a review about these equations).

The sections are organized as follows.

- In the second section, we recall some basic facts about 2D-Toda and the connection with biorthogonal polynomials. We use the approach developed in [13].
- The third section starts with the description of the Toeplitz lattice (see [13]) and shows how to deduce semidiscrete zero-curvature equations for the Ablowitz–Ladik hierarchy.
- In the fourth section, we extend the Toeplitz lattice to the case of block Toeplitz matrices.
- The fifth section gives recursion relations for matrix biorthogonal polynomials on the unit circle; these formulae slightly generalize formulae contained in [19, 22] for matrix orthogonal polynomials on the unit circle.
- In the sixth section, we derive block semidiscrete zero-curvature equations defining the non-Abelian Ablowitz–Ladik hierarchy. As an example we write the non-Abelian analogue of the discrete nonlinear Schrödinger.

2. 2D-Toda; linearization and biorthogonal polynomials

In this section, we recall some basic facts about the 2D-Toda hierarchy as presented in [7]. Moreover we describe the connection with biorthogonal polynomials as originally presented in [13]. We are interested in the semi-infinite case; we start denoting with Λ the shift matrix

$$\Lambda := (\delta_{i+1,j})_{i,j \geqslant 0}.$$

For the transpose we use the notation $\Lambda^{\mathrm{T}} = \Lambda^{-1}$. Then we define two Lax matrices

$$\begin{cases} L_1 := \Lambda + \sum_{i \leqslant 0} a_i^{(1)} \Lambda^i \\ L_2 := a_{-1}^{(2)} \Lambda^{-1} + \sum_{i \geqslant 0} a_i^{(2)} \Lambda^i \end{cases}$$

where $\left\{a_i^{(s)}, s = 1, 2\right\}$ are some diagonal matrices. 2D-Toda equations, expressed in Lax form, arise as compatibility conditions for the following Zakharov–Shabat spectral problem:

$$\begin{cases} L_1\Psi_1 = z\Psi_1 \\ L_2^{\mathrm{T}}\Psi_2^* = z^{-1}\Psi_2^* \\ \partial_{t_n}\Psi_1 = \left(L_1^n\right)_+\Psi_1 \\ \partial_{t_n}\Psi_2^* = -\left(L_1^n\right)_+^{\mathrm{T}}\Psi_2^* \\ \partial_{s_n}\Psi_1 = \left(L_2^n\right)_-\Psi_1 \\ \partial_{s_n}\Psi_2^* = -\left(L_2^n\right)_-^{\mathrm{T}}\Psi_2^*. \end{cases}$$

Here we introduced two infinite sets of times $\{t_i, i \geqslant 0\}$ and $\{s_i, i \geqslant 0\}$. We denoted with N_+ the upper triangular part of a matrix N (including the main diagonal) and with N_- the lower triangular part (excluding the main diagonal). Ψ_1 and Ψ_2^* are semi-infinte column vectors of type

$$\Psi_1(z) = (\Psi_{1,0}(z), \Psi_{1,1}(z), \ldots)^{\mathrm{T}}$$
$$\Psi_2^*(z) = (\Psi_{2,0}^*(z), \Psi_{2,1}^*(z), \ldots)^{\mathrm{T}}.$$

For every k the two expressions $\mathrm{e}^{-\xi(t,z)}\Psi_{1,k}(z)$ and $\mathrm{e}^{-\xi(s,z)}\Psi_{2,k}^*(z^{-1})$ are polynomials in z of order k. Lax equations are written as

$$\partial_{t_n}L_i = \left[\left(L_1^n\right)_+, L_i\right] \qquad \partial_{s_n}L_i = \left[\left(L_2^n\right)_-, L_i\right], \quad i = 1, 2.$$

2D-Toda equations can be linearized as explained in [7]. We start with an initial value matrix $M(0,0) = \{M_{ij}(0,0)\}_{i,j\geqslant 0}$ and we define its time evolution through the equation

$$M(t;s) := \exp(\xi(t,\Lambda))M(0,0)\exp(-\xi(s,\Lambda^{-1})).$$

We assume that there exists a factorization

$$M(0,0) = S_1(0,0)^{-1}S_2(0,0).$$

Here S_1 is lower triangular while S_2 is upper triangular. We assume that both S_1 and S_2 have non-zero elements on the main diagonal and we normalize them in such a way that every element on the main diagonal of S_1 is equal to 1. Moreover we consider values of t and s for which we can write

$$M(t,s) = S_1(t,s)^{-1}S_2(t,s) \tag{1}$$

with S_1 and S_2 having the same properties as above. It can be proven that such factorization exists if and only if all the principal minors of $M(t,s)$ do not vanish. In particular, this condition is satisfied when $M(t,s)$ is the matrix of the moments of a positive-definite measure. Now we denote with $\chi(z)$ the infinite vector $\chi(z) := (1, z, z^2, \ldots)^{\mathrm{T}}$. Wave vectors for 2D-Toda and Lax matrices are constructed in the following way.

Theorem 2.1 ([7]). *The wave vectors*

$$\Psi_1(z) := \exp(\xi(t,z))S_1\chi(z)$$
$$\Psi_2^*(z) := \exp(-\xi(s,z^{-1}))\left(S_2^{-1}\right)^{\mathrm{T}}\chi(z^{-1}).$$

and the two Lax operators $L_1 := S_1\Lambda S_1^{-1}$ *and* $L_2 := S_2\Lambda^{-1}S_2^{-1}$ *satisfy the* 2*D-Toda Zakharov–Shabat spectral problem.*

Proof. We only sketch the proof and make reference to the article [7]. It is clear that the matrix $M(t, s)$ satisfies differential equations

$$\partial_{t_i} M = \Lambda^i M$$
$$\partial_{s_i} M = -M\Lambda^{-i}.$$

Then it is easy to deduce Sato's equations

$$\partial_{t_n} S_1 = -\left(L_1^n\right)_- S_1,$$
$$\partial_{t_n} S_2 = \left(L_1^n\right)_+ S_2,$$
$$\partial_{s_n} S_1 = \left(L_2^n\right)_- S_1,$$
$$\partial_{s_n} S_2 = -\left(L_2^n\right)_+ S_2$$

and Zakharov–Shabat's equations can be deduced from the expression of wave vectors in terms of S_1 and S_2. □

The last thing we need is the link between the factorization of M and biorthogonal polynomials. We introduce a bilinear pairing on the space of polynomials in z defining

$$\langle z^i, z^j \rangle_M := M_{ij}.$$

The following proposition is a direct consequence of (1).

Proposition 2.2 ([13]).

$$q^{(1)} = \left(q_i^{(1)}\right)_{i\geqslant 0} := S_1 \chi(z)$$
$$q^{(2)} = \left(q_i^{(2)}\right)_{i\geqslant 0} := \left(S_2^{-1}\right)^{\mathrm{T}} \chi(z)$$

are biorthonormal polynomials with respect to the pairing $\langle , \rangle_M$*; i.e.*

$$\left\langle q_i^{(1)}, q_j^{(2)} \right\rangle_M = \delta_{ij} \qquad \forall\, i, j \in \mathbb{N}.$$

3. From the Toeplitz lattice hierarchy to the semidiscrete zero-curvature equations for the Ablowitz–Ladik hierarchy

In this section, we briefly recall the reduction from 2D-Toda to the Toeplitz lattice as described in [13]. Then we show how the Ablowitz–Ladik equations are easily obtained from the Toeplitz lattice. Suppose that our initial value $M(0, 0)$ is a Toeplitz matrix; i.e. we have

$$M(0,0) = T(\gamma) = \begin{pmatrix} \gamma^{(0)} & \gamma^{(-1)} & \gamma^{(-2)} & \dots \\ \gamma^{(1)} & \gamma^{(0)} & \gamma^{(-1)} & \dots \\ \gamma^{(2)} & \gamma^{(1)} & \gamma^{(0)} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

for some formal power series $\gamma(z) = \sum_{n\in\mathbb{Z}} \gamma^{(n)} z^n$. Since $\Lambda = T(z^{-1})$ is an upper triangular Toeplitz matrix it follows easily (see for instance [23]) that

$$\begin{aligned} M(t,s) &= \exp(\xi(t,\Lambda)) M(0,0) \exp(-\xi(s,\Lambda^{-1})) \\ &= T(\exp(\xi(t,z^{-1}))\gamma(z)\exp(-\xi(s,z))). \end{aligned}$$

This means that the Toeplitz form is conserved along 2D-Toda flow, hence we deal with a reduction of it. This reduction is called the Toeplitz lattice in [13]. In that article the authors noticed, in the introduction, that this is nothing but the Ablowitz–Ladik hierarchy. Now we will describe how to obtain the original formulation of the Ablowitz–Ladik equations starting from Adler–van Moerbeke's formulation. The key observation is that, in this case, the bilinear pairing $\langle p, q\rangle_M$ between two arbitrary polynomials is given by

$$\langle p, q\rangle_M = \oint p(z)\gamma(z)q^*(z)\frac{\mathrm{d}z}{2\pi \mathrm{i}z}.$$

Here the symbol of integration means that we are taking the residue of the formal series $p(z)\gamma(z)q^*(z)$ and $q^*(z) = q(z^{-1})$. In other words, $q_i^{(1)}$ and $q_j^{(2)}$ are nothing but orthonormal polynomials on the unit circle. We also define monic biorthogonal polynomials

$$p^{(1)} = \big(p_i^{(1)}\big)_{i\geqslant 0} := S_1\chi(z)$$

$$p^{(2)} = \big(p_i^{(2)}\big)_{i\geqslant 0} := h\big(S_2^{-1}\big)^{\mathrm{T}}\chi(z)$$

with $h = \mathrm{diag}(h_0, h_1, h_2, \ldots)$ some diagonal matrix. Given an arbitrary polynomial $q(z)$ of degree n we define its reversed polynomial $\tilde{q}(z) := z^n q^*(z)$ and reflection coefficients

$$x_n := p_n^{(1)}(0) \qquad y_n := p_n^{(2)}(0).$$

We can state the standard recursion relation associated with biorthogonal polynomials on the unit circle. The equation below was already known to Szegő [18]. It has been used in [14] in relation to the theory of integrable equations.

Proposition 3.1. *The following recursion relation holds:*

$$\begin{pmatrix} p_{n+1}^{(1)}(z) \\ \tilde{p}_{n+1}^{(2)}(z)\end{pmatrix} = \mathcal{L}_n \begin{pmatrix} p_{n}^{(1)}(z) \\ \tilde{p}_{n}^{(2)}(z)\end{pmatrix} = \begin{pmatrix} z & x_{n+1} \\ zy_{n+1} & 1\end{pmatrix}\begin{pmatrix} p_{n}^{(1)}(z) \\ \tilde{p}_{n}^{(2)}(z)\end{pmatrix} \tag{2}$$

Using this recursion relation in [13] Adler and van Moerbeke wrote the peculiar form of Lax operators for the Toeplitz reduction.

Proposition 3.2 ([13]). *Lax operators of the Toeplitz lattice are of the following form:*

$$h^{-1}L_1h = \begin{pmatrix} -x_1y_0 & 1-x_1y_1 & 0 & \ldots & \ldots \\ -x_2y_0 & -x_2y_1 & 1-x_2y_2 & 0 & \ldots \\ -x_3y_0 & -x_3y_1 & -x_3y_2 & 1-x_3y_3 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

$$L_2 = \begin{pmatrix} -x_0y_1 & -x_0y_2 & -x_0y_3 & \ldots & \ldots \\ 1-x_1y_1 & -x_1y_2 & -x_1y_3 & \ldots & \ldots \\ 0 & 1-x_2y_2 & -x_2y_3 & \ldots & \ldots \\ \vdots & 0 & 1-x_3y_3 & \ldots & \ldots \\ \vdots & \vdots & 0 & \ldots & \ldots \end{pmatrix}.$$

From this proposition the following corollary follows easily.

Corollary 3.3. *Reflection coefficients satisfy the following equation*

$$\frac{h_{n+1}}{h_n} = 1 - x_{n+1}y_{n+1}.$$

Now we can state the theorem relating the Toeplitz lattice to the original form of the Ablowitz–Ladik hierarchy.

Theorem 3.4. *The Toeplitz lattice flow can be written in the form*

$$\partial_{t_i}\begin{pmatrix} p_n^{(1)}(z) \\ \tilde{p}_n^{(2)}(z) \end{pmatrix} = \mathcal{M}_{t_i,n}\begin{pmatrix} p_n^{(1)}(z) \\ \tilde{p}_n^{(2)}(z) \end{pmatrix} \tag{3}$$

$$\partial_{s_i}\begin{pmatrix} p_n^{(1)}(z) \\ \tilde{p}_n^{(2)}(z) \end{pmatrix} = \mathcal{M}_{s_i,n}\begin{pmatrix} p_n^{(1)}(z) \\ \tilde{p}_n^{(2)}(z) \end{pmatrix} \tag{4}$$

for some matrices $\mathcal{M}_{t_i,n}, \mathcal{M}_{s_i,n}$ *depending on* $\{x_j, y_j, z\}$.

Proof. We prove it for the set of times denoted by t. We denote $d([z]) = \operatorname{diag}(1, z, z^2, z^3, \ldots)$. We have the identities

$$\Psi_1(z) = \exp(\xi(t,z))p^{(1)}(z)$$
$$\Psi_2^*(z) = h^{-1}d([z^{-1}])\exp(-\xi(s,z^{-1}))\tilde{p}^{(2)}(z)$$

giving the following time evolution for orthogonal polynomials:

$$\partial_{t_i}p^{(1)}(z) = -z^i p^{(1)}(z) + \left(L_1^i\right)_+ p^{(1)}(z)$$
$$\partial_{t_i}\tilde{p}^{(2)}(z) = -hd([z])\left(L_1^i\right)_{++}^{\mathrm{T}}h^{-1}d([z^{-1}])\tilde{p}^{(2)}(z).$$

Here $\left(L_1^i\right)_{++}$ denotes the strictly upper diagonal part of L_1^n. The formulae above are obtained from a straightforward computation and using the fact, proven in [13], that

$$\partial_{t_i}\log(h_n) = \left(L_1^i\right)_{nn}.$$

Hence we have that, for every k, $\partial_{t_i}p_k^{(1)}$ is a linear combination of $\left\{p_k^{(1)}, p_{k+1}^{(1)}, p_{k+2}^{(1)}, \ldots\right\}$ with coefficients in $\mathbb{C}[x_j, y_j]$. In the same way, for every k, $\partial_{t_i}\tilde{p}_k^{(2)}$ is a linear combination of $\left\{\tilde{p}_k^{(2)}, \tilde{p}_{k-1}^{(2)}, \tilde{p}_{k-2}^{(2)}, \ldots\right\}$ with coefficients in $\mathbb{C}[x_j, y_j]$. Using the recursion relation (2) and its inverse

$$\begin{pmatrix} p_n^{(1)}(z) \\ \tilde{p}_n^{(2)}(z) \end{pmatrix} = \mathcal{L}_n^{-1}\begin{pmatrix} p_{n+1}^{(1)}(z) \\ \tilde{p}_{n+1}^{(2)}(z) \end{pmatrix} = \frac{h_n}{h_{n+1}}\begin{pmatrix} z^{-1} & -z^{-1}x_{n+1} \\ -y_{n+1} & 1 \end{pmatrix}\begin{pmatrix} p_{n+1}^{(1)}(z) \\ \tilde{p}_{n+1}^{(2)}(z) \end{pmatrix}$$

we can obtain the desired matrices $\mathcal{M}_{t_i,n}$. □

Corollary 3.5 (Ablowitz–Ladik semidiscrete zero-curvature equations). *The matrices* $\mathcal{L}_n$ *satisfy the following time evolution:*

$$\partial_{t_i}\mathcal{L}_n = \mathcal{M}_{t_i,n+1}\mathcal{L}_n - \mathcal{L}_n\mathcal{M}_{t_i,n} \tag{5}$$

$$\partial_{s_i}\mathcal{L}_n = \mathcal{M}_{s_i,n+1}\mathcal{L}_n - \mathcal{L}_n\mathcal{M}_{s_i,n}. \tag{6}$$

Proof. These equations are nothing but the compatibility conditions of recursion relation (2) with time evolution (3) and (4). □

Remark 3.6. Actually our Lax operator $\mathcal{L}_n$ is slightly different from the Lax operator L_n used in [1] by Ablowitz and Ladik and written in the introduction above. Nevertheless, as shown in [16], these two Lax operators are linked through a simple change of spectral parameter.

Example 3.7 (the first flows; discrete nonlinear Schrödinger). The first matrices $\mathcal{M}_{t_i,n}$ and $\mathcal{M}_{s_i,n}$ are easily computed. We have

$$\partial_{t_1} p_k^{(1)} = -zp_k^{(1)} - x_{k+1} y_k p_k^{(1)} + p_{k+1}^{(1)} = -x_{k+1} y_k p_k^{(1)} + x_{k+1} \tilde{p}_k^{(2)}$$

$$\partial_{t_1} \tilde{p}_k^{(2)} = -z \frac{h_{k+1}}{h_k} \tilde{p}_{k-1}^{(2)} = z y_k p_k^{(1)} - z \tilde{p}_k^{(2)}$$

giving immediately

$$\mathcal{M}_{t_1,k} = \begin{pmatrix} -x_{k+1} y_k & x_{k+1} \\ z y_k & -z \end{pmatrix}$$

An analogue computation for s_1 can be easily done obtaining

$$\mathcal{M}_{s_1,k} = \begin{pmatrix} z^{-1} & -z^{-1} x_k \\ -y_{k+1} & x_k y_{k+1} \end{pmatrix}.$$

We can already write, with t_1 and s_1, the well-known integrable discretization of nonlinear Schrödinger. We just need two more trivial rescaling times introduced with substitutions

$$p^{(1)} \longmapsto \exp(t_0) p^{(1)}$$
$$\tilde{p}^{(2)} \longmapsto \exp(-s_0) \tilde{p}^{(2)}$$

and corresponding to matrices

$$\mathcal{M}_{t_0,k} := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad \mathcal{M}_{s_0,k} := \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}.$$

Now we can construct the matrix

$$\mathcal{M}_{\tau,k} = \mathcal{M}_{t_1,k} + \mathcal{M}_{s_1,k} - \mathcal{M}_{t_0,k} - \mathcal{M}_{s_0,k}$$
$$= \begin{pmatrix} z^{-1} - 1 - x_{k+1} y_k & x_{k+1} - z^{-1} x_k \\ z y_k - y_{k+1} & x_k y_{k+1} + 1 - z \end{pmatrix}$$

associated with the time $\tau = t_1 + s_1 - t_0 - s_0$ so that semidiscrete zero-curvature equation

$$\partial_\tau \mathcal{L}_k = \mathcal{M}_{\tau,k+1} \mathcal{L}_k - \mathcal{L}_k \mathcal{M}_{\tau,k}$$

is equivalent to the system

$$\begin{cases} \partial_\tau x_k = x_{k+1} - 2x_k + x_{k-1} - x_k y_k (x_{k+1} + x_{k-1}) \\ \partial_\tau y_k = -y_{k+1} + 2y_k - y_{k-1} + x_k y_k (y_{k+1} + y_{k-1}). \end{cases} \tag{7}$$

This is exactly the complexified version of the discrete nonlinear Schrödinger. Rescaling $\tau \mapsto \mathrm{i}\tau$ and imposing $y_k = \pm x_k^*$ we obtain

$$-\mathrm{i}\partial_\tau x_k = x_{k+1} - 2x_k + x_{k-1} \mp \|x_k\|^2 (x_{k+1} + x_{k-1}). \tag{8}$$

4. Toda flow for block Toeplitz matrices and the related Lax operators

Now we generalize the Toeplitz lattice's equations to the block case. We start with a matrix-valued formal series

$$\gamma(z) = \sum_{k \in \mathbb{Z}} \gamma^{(k)} z^k.$$

Here every element $\gamma^{(k)}$ is a $n \times n$ matrix. Then we define its time evolution as

$$\gamma(t, s; z) := \exp(-\xi(s, z^{-1} I)) \gamma(z) \exp(\xi(t, z I)).$$

where I is the $n \times n$ identity matrix. Differently from the scalar case we do not consider just one Toeplitz matrix but the two block Toeplitz matrices, right and left, given by

$$T^r(\gamma) := \begin{pmatrix} \gamma^{(0)} & \gamma^{(-1)} & \gamma^{(-2)} & \dots \\ \gamma^{(1)} & \gamma^{(0)} & \gamma^{(-1)} & \dots \\ \gamma^{(2)} & \gamma^{(1)} & \gamma^{(0)} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$T^l(\gamma) := \begin{pmatrix} \gamma^{(0)} & \gamma^{(1)} & \gamma^{(2)} & \dots \\ \gamma^{(-1)} & \gamma^{(0)} & \gamma^{(1)} & \dots \\ \gamma^{(-2)} & \gamma^{(-1)} & \gamma^{(0)} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

In this way, we obtain the following linear time evolution for our block Toeplitz matrices (in the following we will omit the symbol γ):

$$\partial_{t_i} T^l = \Lambda^i T^l \qquad \partial_{t_i} T^r = T^r \Lambda^{-i} \tag{9}$$

$$\partial_{s_i} T^l = -T^l \Lambda^{-i} \qquad \partial_{s_i} T^r = -\Lambda^i T^r \tag{10}$$

where, in this case, we have $\Lambda = T^r(z^{-1}I)$. Then we assume that there exist two factorizations

$$T^l = S_1^{-1} S_2 \qquad T^r = Z_2 Z_1^{-1}.$$

Here S_1, Z_2 are block-lower triangular while S_2, Z_1 are block-upper triangular. We assume that all these matrices have non-degenerate blocks on the main diagonal (i.e. these blocks must have non-zero determinants). Normalizations are chosen in such a way that every element on the main block-diagonal of S_1 and Z_2 is equal to the identity matrix I. As we did previously we assume that these conditions hold when every time is equal to 0 and then we consider just the values of t and s for which these conditions still hold. In the matrix case, we can define two bilinear pairings given by the following definition.

Definition 4.1.

$$\langle P, Q\rangle_r := \oint P^*(z)\gamma(z)Q(z)\frac{\mathrm{d}z}{2\pi \mathrm{i}z} \qquad \langle P, Q\rangle_l := \oint P(z)\gamma(z)Q^*(z)\frac{\mathrm{d}z}{2\pi \mathrm{i}z}$$

where P and Q are two arbitrary matrix polynomials and $P^*(z) := (P(z^{-1}))^{\mathrm{T}}$.

Our two factorizations give exactly biorthonormal polynomials for $\langle , \rangle_r$ and $\langle , \rangle_l$. In the following we denote $\chi(z) := (I, zI, z^2I, z^3I, \dots)^{\mathrm{T}}$.

Proposition 4.2.

$$Q^{(1)l} := \begin{pmatrix} Q_0^{(1)l} \\ Q_1^{(1)l} \\ \vdots \end{pmatrix} = S_1 \chi(z) \tag{11}$$

$$Q^{(2)l} := \begin{pmatrix} Q_0^{(2)l} \\ Q_1^{(2)l} \\ \vdots \end{pmatrix} = \left(S_2^{-1}\right)^{\mathrm{T}} \chi(z) \tag{12}$$

$$Q^{(1)r} := \begin{pmatrix} Q_0^{(1)r} & Q_1^{(1)r} & \dots \end{pmatrix} = \chi(z)^{\mathrm{T}} Z_1 \tag{13}$$

$$Q^{(2)r} := \begin{pmatrix} Q_0^{(2)r} & Q_1^{(2)r} & \dots \end{pmatrix} = \chi(z)^{\mathrm{T}} \left(Z_2^{-1}\right)^{\mathrm{T}} \tag{14}$$

are the biorthonormal polynomials associated with the pairing $\langle , \rangle_l$ and $\langle , \rangle_r$. Hence for every i, j we have

$$\left\langle Q_i^{(1)l}, Q_j^{(2)l} \right\rangle_l = \delta_{ij} \qquad \left\langle Q_i^{(2)r}, Q_j^{(1)r} \right\rangle_r = \delta_{ij}.$$

Proof. We only prove, as an example, the proposition for the right polynomials. On the other hand, the one for left polynomials is identical to the usual proof for 2D-Toda. We have

$$\left(\left\langle Q_i^{(2)r}, Q_j^{(1)r} \right\rangle_r\right)_{i,j\geqslant 0} = \left(\sum_{k,l\geqslant 0} \left(Z_2^{-1}\right)_{ki} \langle z^k I, z^l I\rangle_r (Z_1)_{lj} \right)$$

$$= Z_2^{-1} T^l Z_1 = I \Longleftrightarrow T^l = Z_2 Z^{-1}.$$

(It should be noted that, in this case, subscripts of type $(Z_1)_{ij}$ denote the block in position (i, j) and not the element (i, j).) □

Now we can write the related Sato's equations for S_i and Z_i. It is convenient to introduce the following Lax operators.

Definition 4.3.

$$L_1 := S_1 \Lambda S_1^{-1}, \quad L_2 := S_2 \Lambda^{-1} S_2^{-1} \tag{15}$$

$$R_1 := Z_1^{-1} \Lambda^{-1} Z_1, \quad R_2 := Z_2^{-1} \Lambda Z_2. \tag{16}$$

Proposition 4.4. *The following Sato's equations are satisfied:*

$$\partial_{t_n} S_1 = -\left(L_1^n\right)_- S_1 \qquad \partial_{t_n} Z_1 = -Z_1\left(R_1^n\right)_+, \tag{17}$$

$$\partial_{t_n} S_2 = \left(L_1^n\right)_+ S_2 \qquad \partial_{t_n} Z_2 = Z_2\left(R_1^n\right)_-, \tag{18}$$

$$\partial_{s_n} S_1 = \left(L_2^n\right)_- S_1 \qquad \partial_{s_n} Z_1 = Z_1\left(R_2^n\right)_+, \tag{19}$$

$$\partial_{s_n} S_2 = -\left(L_2^n\right)_+ S_2 \qquad \partial_{s_n} Z_2 = -Z_2\left(R_2^n\right)_-. \tag{20}$$

Proof. We will only prove, as an example, the equations involving the t-derivative of Z_1 and Z_2. We assume, as an ansatz, that we have

$$\partial_{t_n} Z_1 = Z_1 A$$
$$\partial_{t_n} Z_2 = Z_2 B$$

for some matrices A and B. Then exploiting the time evolution of T^r we can write

$$T^r \Lambda^{-n} = \partial_{t_n} T^r = \partial_{t_n}\left(Z_2 Z_1^{-1}\right)$$
$$= Z_2 B Z_1^{-1} - Z_2 Z_1^{-1} Z_1 A Z_1^{-1} = Z_2(B - A) Z_1^{-1}$$

hence we must have $(B - A) Z_1^{-1} = Z_1^{-1} \Lambda^{-n}$. We rewrite it as

$$B - A = Z_1^{-1} \Lambda^{-n} Z_1 = R_1^n.$$

Moreover, we have that B must be strictly block-lower triangular and A block-upper triangular. This is because Z_1 is block-upper triangular and Z_2 is block-lower triangular with constant entries on the main diagonal. Hence we conclude that $A = -\left(R_1^n\right)_+$ and $B = \left(R_1^n\right)_-$. □

Now it is just a matter of trivial computations to write down the corresponding Lax equations for L_i and R_i.

Proposition 4.5. *The following Lax equations are satisfied:*

$$\partial_{t_n} L_i = \left[\left(L_1^n\right)_+, L_i\right], \qquad \partial_{t_n} R_i = \left[R_i, \left(R_1^n\right)_-\right], \tag{21}$$

$$\partial_{s_n} L_i = \left[\left(L_2^n\right)_-, L_i\right], \qquad \partial_{s_n} R_i = \left[R_i, \left(R_2^n\right)_+\right]. \tag{22}$$

The definition of our Lax operators will give us eigenvalue equations for suitably defined wave vectors.

Definition 4.6.

$$\Psi_1(z) := \exp(\xi(t, zI))S_1\chi(z) \tag{23}$$

$$\Phi_1(z) := \exp(\xi(t, zI))[\chi(z)]^{\mathrm{T}} Z_1 \tag{24}$$

$$\Psi_2^*(z) := \exp(-\xi(s, z^{-1}I))\left(S_2^{-1}\right)^{\mathrm{T}}\chi(z^{-1}) \tag{25}$$

$$\Phi_2^*(z) := \exp(-\xi(s, z^{-1}I))\chi(z^{-1})^{\mathrm{T}}\left(Z_2^{-1}\right)^{\mathrm{T}}. \tag{26}$$

Proposition 4.7. *The following equations hold true:*

$$L_1\Psi_1(z) = z\Psi_1(z), \qquad \Phi_1(z)R_1 = z\Phi_1(z), \tag{27}$$

$$L_2^{\mathrm{T}}\Psi_2^*(z) = z^{-1}\Psi_2^*(z), \qquad \Phi_2^*(z)R_2^{\mathrm{T}} = z^{-1}\Phi_2^*(z). \tag{28}$$

Proof. We will only prove the last equation, all the other ones are proved in a similar way. From its very definition we have

$$\begin{aligned}\Phi_2^*(z)R_2^{\mathrm{T}} = z^{-1}I\Phi_2^*(z) &\Longleftrightarrow [\chi(z^{-1})]^{\mathrm{T}}\left(Z_2^{-1}\right)^{\mathrm{T}}R_2^{\mathrm{T}} = z^{-1}[\chi(z^{-1})]^{\mathrm{T}}\left(Z_2^{-1}\right)^{\mathrm{T}}\\ &= [\chi(z^{-1})]^{T}\Lambda^{-1}\left(Z_2^{-1}\right)^{\mathrm{T}} \Longleftrightarrow R_2^{\mathrm{T}} = Z_2^{\mathrm{T}}\Lambda^{-1}\left(Z_2^{-1}\right)^{\mathrm{T}} \Longleftrightarrow R_2 = Z_2^{-1}\Lambda Z_2.\end{aligned}$$

□

The proof of the following proposition is straightforward.

Proposition 4.8. *The Lax equations (21) and (22) are the compatibility conditions of the eigenvalue equations (27) and (28) with the following equations:*

$$\partial_{t_n}\Psi_1 = \left(L_1^n\right)_+\Psi_1 \qquad \partial_{t_n}\Phi_1 = \Phi_1\left(R_1^n\right)_- \tag{29}$$

$$\partial_{s_n}\Psi_1 = \left(L_2^n\right)_-\Psi_1 \qquad \partial_{s_n}\Phi_1 = \Phi_1\left(R_2^n\right)_+ \tag{30}$$

$$\partial_{t_n}\Psi_2^* = -\left(L_{1+}^n\right)^{\mathrm{T}}\Psi_2^* \qquad \partial_{t_n}\Phi_2^* = -\Phi_2^*\left(R_{1-}^n\right)^{\mathrm{T}} \tag{31}$$

$$\partial_{s_n}\Psi_2^* = -\left(L_{2-}^n\right)^{\mathrm{T}}\Psi_2^* \qquad \partial_{s_n}\Phi_2^* = -\Phi_2^*\left(R_{2+}^n\right)^{\mathrm{T}}. \tag{32}$$

Remark 4.9. Actually our Lax equations as well as equations (27)–(32) can be deduced from the equations of multicomponent 2D-Toda [10].

5. Recursion relations for matrix biorthogonal polynomials on the unit circle

In order to generalize the scalar theory we have to construct an analogue of the recursion relation given by proposition 3.1. Recursion relations for the matrix orthogonal polynomial on the unit circle are already known, see [19, 22]. Here we slightly generalize to the case of matrix biorthogonal polynomials on the unit circle.

We define the following important $n \times n$ matrices.

Definition 5.1.

$$h^r_N := \mathrm{SC}\big(T^r_{N+1}\big) \qquad h^l_N := \mathrm{SC}\big(T^l_{N+1}\big)$$

where SC denotes the $n \times n$ Schur complement of a block matrix with respect to the upper left block (see for instance [21]). For example we have

$$\mathrm{SC}\big(T^r_{N+1}\big) = \gamma^{(0)} - (\gamma^{(N)} \quad \dots \quad \dots \quad \gamma^{(1)})T^{-r}_N \begin{pmatrix} \gamma^{(-N)} \\ \dots \\ \dots \\ \gamma^{(-1)} \end{pmatrix}$$

(here and below $T^{-r}_N := \big(T^r_N\big)^{-1}$ and similarly for h^r_N, h^l_N and T^l_N).

Proposition 5.2. *Monic biorthogonal polynomials such that*

$$\big\langle P^{(2)r}_k, P^{(1)r}_j\big\rangle_r = \delta_{kj} h^r_k \qquad \big\langle P^{(1)l}_k, P^{(2)l}_j\big\rangle_l = \delta_{kj} h^l_k.$$

are given by the following formulae:

$$P^{(1)r}_N = \mathrm{SC}\begin{pmatrix} \gamma^{(0)} & \dots & \dots & \gamma^{(-N+1)} & \gamma^{(-N)} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \gamma^{(N-1)} & \dots & \dots & \gamma^{(0)} & \gamma^{(-1)} \\ I & zI & \dots & z^{N-1}I & z^N I \end{pmatrix}$$

$$\big(P^{(2)r}_N\big)^{\mathrm{T}} = \mathrm{SC}\begin{pmatrix} \gamma^{(0)} & \dots & \dots & \gamma^{(-N+1)} & I \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \gamma^{(N-1)} & \dots & \dots & \gamma^{(0)} & z^{N-1}I \\ \gamma^{(N)} & \dots & \dots & \gamma^{(1)} & z^N I \end{pmatrix}$$

$$P^{(1)l}_N = \mathrm{SC}\begin{pmatrix} \gamma^{(0)} & \dots & \dots & \gamma^{(N-1)} & I \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \gamma^{(-N+1)} & \dots & \dots & \gamma^{(0)} & z^{N-1}I \\ \gamma^{(-N)} & \dots & \dots & \gamma^{(-1)} & z^N I \end{pmatrix}$$

$$\big(P^{(2)l}_N\big)^{T} = \mathrm{SC}\begin{pmatrix} \gamma^{(0)} & \dots & \dots & \gamma^{(N-1)} & \gamma^{(1)} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \gamma^{(-N+1)} & \dots & \dots & \gamma^{(0)} & \gamma^{(N-1)} \\ I & zI & \dots & z^{N-1}I & z^N I \end{pmatrix}.$$

Proof. We will only prove the first formula, the second one is proved similarly. First of all we have $\forall\, 0 \leqslant m \leqslant N-1$

$$\left\langle z^m I, P_N^{(1)r}(z)\right\rangle_r = \oint z^{-m}\gamma(z)\left(z^N I - \begin{pmatrix} I & \dots & \dots & z^{N-1} I\end{pmatrix} T_N^{-r}\begin{pmatrix}\gamma^{(-N)}\\ \dots \\ \dots \\ \gamma^{(-1)}\end{pmatrix}\right)\frac{\mathrm{d}z}{2\pi \mathrm{i} z}$$

$$= \gamma^{(m-N)} - \gamma^{(m-N)} = 0.$$

In the same way $\forall\, 0 \leqslant m \leqslant N-1$

$$\left\langle P_N^{(2)r}(z), z^m I\right\rangle_r = \oint \left(z^{-N} I - (\gamma^{(N)} \quad \dots \quad \dots \quad \gamma^{(1)}) T_N^{-r}\begin{pmatrix} I \\ \dots \\ \dots \\ z^{-N+1} I\end{pmatrix}\right)\gamma(z) z^m \frac{\mathrm{d}z}{2\pi \mathrm{i} z}$$

$$= \gamma^{(N-m)} - \gamma^{(N-m)} = 0.$$

Finally

$$\left\langle P_N^{(2)r}(z), P_N^{(1)r}(z)\right\rangle_r = \left\langle z^N I, P_N^{(1)r}\right\rangle_r = \gamma^{(0)} - (\gamma^{(N)} \quad \dots \quad \dots \quad \gamma^{(1)}) T_N^{-r}\begin{pmatrix}\gamma^{(-N)}\\ \dots \\ \dots \\ \gamma^{(-1)}\end{pmatrix} = h_N^r.$$

This completes the proof of the first formula, the second one is proved similarly. □

Remark 5.3. Note that imposing

$$Q_k^{(1)l} := P_k^{(1)l} \qquad Q_k^{(2)l} := \left(h_k^{-l}\right)^{\mathrm{T}} P_k^{(2)l}$$
$$Q_k^{(1)r} := P_k^{(1)r}\left(h_k^{-r}\right) \qquad Q_k^{(2)r} := P_k^{(2)r}.$$

we obtain biorthonormal polynomials.

Now we will write a long list of relations among these polynomials and reflection coefficients. Given any matrix polynomial $Q(z)$ of degree n we define the associated reversed polynomial as

$$\tilde{Q}(z) = z^n Q^*(z).$$

The reflection coefficients are defined as follows:

$$x_N^l := P_N^{(1)l}(0), \qquad x_N^r := P_N^{(1)r}(0),$$
$$y_N^l := \left(P_N^{(2)l}(0)\right)^{\mathrm{T}}, \qquad y_N^r := \left(P_N^{(2)r}(0)\right)^{\mathrm{T}}.$$

Proposition 5.4. *The following formulae hold true:*

$$P_{N+1}^{(1)l} - z P_N^{(1)l} = x_{N+1}^l \tilde{P}_N^{(2)r}, \tag{33}$$

$$\tilde{P}_{N+1}^{(2)r} - \tilde{P}_N^{(2)r} = z y_{N+1}^r P_N^{(1)l}, \tag{34}$$

$$P_{N+1}^{(1)r} - z P_N^{(1)r} = \tilde{P}_N^{(2)l} x_{N+1}^r, \tag{35}$$

$$\tilde{P}_{N+1}^{(2)l} - \tilde{P}_N^{(2)l} = z P_N^{(1)r} y_{N+1}^l, \tag{36}$$

$$P_{N+1}^{(1)r} = z P_N^{(1)r}\left(I - y_{N+1}^l x_{N+1}^r\right) + \tilde{P}_{N+1}^{(2)l} x_{N+1}^r, \tag{37}$$

$$P_{N+1}^{(1)l} = z\left(I - x_{N+1}^l y_{N+1}^r\right) P_N^{(1)l} + x_{N+1}^l \tilde{P}_{N+1}^{(2)l}, \tag{38}$$

$$\tilde{P}^{(2)r}_{N+1} = \left(I - y^r_{N+1} x^l_{N+1}\right)\tilde{P}^{(2)r}_N + y^r_{N+1} P^{(1)l}_{N+1}, \tag{39}$$

$$\tilde{P}^{(2)l}_{N+1} = \tilde{P}^{(2)l}_N\left(I - x^r_{N+1} y^l_{N+1}\right) + P^{(1)r}_{N+1} y^l_{N+1}, \tag{40}$$

$$x^l_N h^r_N = h^l_N x^r_N, \tag{41}$$

$$y^r_N h^l_N = h^r_N y^l_N, \tag{42}$$

$$h^{-r}_N h^r_{N+1} = I - y^l_{N+1} x^r_{N+1}, \tag{43}$$

$$h^l_{N+1} h^{-l}_N = I - x^l_{N+1} y^r_{N+1}. \tag{44}$$

Proof. The first four formulae are proved observing, for instance for the first case, that $\forall\, 1 \leqslant i \leqslant N$ we have

$$0 = \left\langle P^{(1)l}_{N+1} - z P^{(1)l}_N, z^i I\right\rangle_l = \left\langle \tilde{P}^{(2)r}_N, z^i I\right\rangle_l$$

so that $P^{(1)l}_{N+1} - z P^{(1)l}_N$ and $\tilde{P}^{(2)r}_N$ must be proportional. Setting $z = 0$ you also find the constant of proportionality. In particular, when proving (34) and (36), we find a formula and then we have to take the reversed one. Equation (37) is proved substituting (36) into (35) and similarly for (38), (39), (40).

Equations (41) and (42) are proven respectively observing that we have

$$\left\langle \tilde{P}^{(1)l}_N, P^{(1)r}_N\right\rangle_r = \left\langle P^{(1)l}_N, \tilde{P}^{(1)r}_N\right\rangle_l$$

and

$$\left\langle P^{(2)r}_N, \tilde{P}^{(2)l}_N\right\rangle_r = \left\langle \tilde{P}^{(2)r}_N, P^{(2)l}_N\right\rangle_l$$

and then doing explicit computations. Finally (43) is obtained rewriting (37) as

$$\frac{P^{(1)r}_{N+1}}{z^{N+1}} = \frac{P^{(1)r}_N}{z^N}\left(I - y^l_{N+1} x^r_{N+1}\right) + \left(P^{(2)l}_{N+1}\right)^* x^r_{N+1},$$

multiplying from the left for $P^{(1)l}_N \gamma$ and then taking the residue. Equation (44) is proved similarly. □

Now we define two sets of block matrices $\left\{\mathcal{L}^r_N\right\}_{N\geqslant 0}$ and $\left\{\mathcal{L}^l_N\right\}_{N\geqslant 0}$. They will have, in the matrix case, the same role played by $\{\mathcal{L}_n\}_{n\geqslant 0}$ in the scalar case.

Definition 5.5.

$$\mathcal{L}^l_N := \begin{pmatrix} zI & x^l_{N+1} \\ z y^r_{N+1} & I \end{pmatrix} \tag{45}$$

$$\mathcal{L}^r_N := \begin{pmatrix} zI & z y^l_{N+1} \\ x^r_{N+1} & I \end{pmatrix}. \tag{46}$$

Corollary 5.6. *The following block matrices recursion relations are satisfied:*

$$\begin{pmatrix} P^{(1)l}_{N+1} \\ \tilde{P}^{(2)r}_{N+1} \end{pmatrix} = \mathcal{L}^l_N \begin{pmatrix} P^{(1)l}_N \\ \tilde{P}^{(2)r}_N \end{pmatrix} \tag{47}$$

$$\begin{pmatrix} P^{(1)r}_{N+1} & \tilde{P}^{(2)l}_{N+1} \end{pmatrix} = \begin{pmatrix} P^{(1)r}_N & \tilde{P}^{(2)l}_N \end{pmatrix} \mathcal{L}^r_N. \tag{48}$$

Proof. These are nothing but (33)–(36). □

6. Explicit expressions for Lax operators and related semidiscrete zero-curvature equations

Using our recursion relations we want to find explicit expressions for L_i and R_i in terms of our reflection coefficients x_k^l, x_k^r, y_k^l, y_k^r. First of all we underline a remarkable symmetry that will allow us to reduce the amount of our computations. Doing the following three substitutions:

$$z \mapsto z^{-1}$$
$$t \mapsto -s$$
$$s \mapsto -t$$

we immediately obtain the following proposition.

Proposition 6.1. *Under the symmetry above the dressings, the orthogonal polynomials, the Lax operators and the reflection coefficients change as follows:*

$$T^r \mapsto T^l \qquad T^l \mapsto T^r$$
$$S_1 \mapsto Z_2^{-1} \qquad S_2 \mapsto Z_1^{-1}$$
$$L_1 \mapsto R_2 \qquad L_2 \mapsto R_1$$
$$Q^{(1)l} \mapsto (Q^{(2)r})^* \qquad Q^{(2)l} \mapsto (Q^{(1)r})^*$$
$$P^{(1)l} \mapsto (P^{(2)r})^* \qquad P^{(2)l} \mapsto (P^{(2)r})^*$$
$$x^l \mapsto y^r \qquad y^l \mapsto x^r$$
$$h_k^l \mapsto h_k^r.$$

Hence we can write only the left theory and we will have the right one as well. Actually every computation made above for the right theory can be deduced from the left theory and this symmetry will be called $t{-}s$ symmetry in the following. In the theorem below the symbol $\prod_{j=N+2}^{M-}$ means that the terms in the product must be written from the smallest index to the biggest, going from left to right. The symbol $\prod_{j=N+2}^{M+}$ means that the product must be taken in the opposite direction.

Theorem 6.2 (Lax operators for block Toeplitz lattice). *Lax operators L_i and R_i are expressed in terms of reflection coefficients according to the following formulae:*

$\forall\, N > M \geqslant -1$

$$(L_1)_{N,M+1} = -x_{N+1}^l \left(\prod_{j=N+2}^{M-} \left(I - y_j^r x_j^l\right) \right) y_{M+1}^r \tag{49}$$

$$(R_2)_{N,M+1} = -y_{N+1}^r \left(\prod_{j=N+2}^{M-} \left(I - x_j^l y_j^r\right) \right) x_{M+1}^l \tag{50}$$

$$(L_2)_{M+1,N} = -h_{M+1}^{-l} x_{M+1}^r \left(\prod_{j=N+2}^{M+} \left(I - y_j^l x_j^r\right) \right) y_{N+1}^l h_N^l \tag{51}$$

$$(R_1)_{M+1,N} = -h_{M+1}^{-r} y_{M+1}^l \left(\prod_{j=N+2}^{M+} \left(I - x_j^r y_j^l\right) \right) x_{N+1}^r h_N^r. \tag{52}$$

Moreover

$$(L_1)_{N,N+1} = (R_2)_{N,N+1} = I \tag{53}$$

$$(L_2)_{N+1,N} = h^l_{N+1}h^{-l}_N \quad (R_1)_{N+1,N} = h^r_{N+1}h^{-r}_N. \tag{54}$$

Proof. Equations (53) and (54) follow trivially from the expressions of dressings S_i, Z_i. In fact, because of our normalization, we have that S_1 and Z_2 are equal to the identity matrix plus a strictly lower triangular matrix which is what is stated in (53). Equation (54) is obtained by observing that the block on the main diagonal of S_2 and Z_1 is given respectively by matrices h^l_k and h^r_k as follows from proposition 4.2 and remark 5.3. Now let us begin with (49) and (50); the important point is that we have

$$\Psi_1 = \exp(\xi(t, zI))P^{(1)l}.$$

Hence as done in [13] we can find that $\forall\, N > M \geqslant -1$ we have

$$(L_1)_{N,M+1} = -x^l_{N+1}h^r_N h^{-r}_{M+1}y^r_{M+1}. \tag{55}$$

In fact $\forall\, N > M \geqslant -1$

$$\begin{aligned}\left\langle P^{(1)l}_{N+1} - zP^{(1)l}_N, P^{(2)l}_{M+1} - zP^{(2)l}_M\right\rangle_l &= -\left\langle zP^{(1)l}_N, P^{(2)l}_{M+1}\right\rangle_l \\ &= -\left\langle P^{(1)l}_{N+1} + \cdots + (L_1)_{N,M+1}P^{(1)l}_{M+1} + \cdots, P^{(2)l}_{M+1}\right\rangle_l = -(L_1)_{N,M+1}h^l_{M+1}.\end{aligned}$$

On the other hand, using recursion relations, we also have $\forall N \geqslant M \geqslant -1$

$$\begin{aligned}\left\langle P^{(1)l}_{N+1} - zP^{(1)l}_N, P^{(2)l}_{M+1} - zP^{(2)l}_M\right\rangle_l &= \left\langle x^l_{N+1}\tilde{P}^{(2)r}_N, \left(y^l_{M+1}\right)^{\mathrm{T}}\tilde{P}^{(1)r}_M\right\rangle_l \\ &= x^l_{N+1}\left(\oint z^{N-M}\left(P^{(2)r}_N\right)^*\gamma(z)P^{(1)r}_M\frac{\mathrm{d}z}{2\pi \mathrm{i}z}\right)y^l_{M+1} \\ &= x^l_{N+1}\left\langle P^{(2)r}_N, z^{N-M}P^{(1)r}_M\right\rangle_r y^l_{M+1} = x^l_{N+1}h^r_N y^l_{M+1}\end{aligned}$$

and comparing them we find (55). Now we use t–s symmetry to simplify this expression. We obtain

$$(R_2)_{N,M+1} = -y^r_{N+1}h^l_N h^{-l}_{M+1}x^l_{M+1}.$$

Now we apply the recursion (44) to the piece $h^l_N h^{-l}_{M+1}$ several times and we get (50). Equation (49) is obtained using t–s symmetry. For (51) and (52) we start by defining $\tilde{R}_1$ such that $zP^{(1)r} = P^{(1)r}\tilde{R}_1$; then we will have $R_1 = h^r\tilde{R}_1h^{-r}$ and computations for $\tilde{R}_1$ are carried out similarly as for L_1. □

Remark 6.3. Our equations (49), (50), (51) and (52) extend to the matrix biorthogonal setting the equations written in [22] for matrix orthogonal polynomials (see equations (4.2) and (4.3)). In that article, the properties of M are applied to study some problems in computational mathematics (multivariate time series analysis and multichannel signal processing) and no relation is established with the Lax theory and integrable systems.

The theorem above describes the block-analogue of the Toeplitz lattice; we are now in the position to prove the analogue of theorem 3.4.

Theorem 6.4. *Block Toeplitz lattice flow can be written in the form*

$$\partial_{t_i/s_i}\begin{pmatrix} P^{(1)l}_N(z) \\ \tilde{P}^{(2)r}_N(z)\end{pmatrix} = \mathcal{M}^l_{t_i/s_i,N}\begin{pmatrix} P^{(1)l}_N(z) \\ \tilde{P}^{(2)r}_N(z)\end{pmatrix} \tag{56}$$

$$\partial_{t_i/s_i}\left(P_N^{(1)r}(z) \quad \tilde{P}_N^{(2)l}(z)\right) = \left(P_N^{(1)r}(z) \quad \tilde{P}_N^{(2)l}(z)\right)\mathcal{M}^r_{t_i/s_i,N} \tag{57}$$

for some block matrices $\mathcal{M}^r_{t_i,N}, \mathcal{M}^r_{s_i,N}, \mathcal{M}^l_{t_i,N}, \mathcal{M}^l_{s_i,N}$ depending on the matrices $\left\{x_j^l, y_j^l, x_j^r, y_j^r\right\}$ and the spectral parameter z.

Proof. As we did for the scalar case we prove it only for t times. The relevant equations linking biorthogonal polynomials with wave vectors are

$$\Psi_1(z) = \exp(\xi(t, zI))P^{(1)l}(z),$$
$$\Phi_2^*(z) = \exp(-\xi(s, z^{-1}I))(\tilde{P}^{(2)r})^{\mathrm{T}}d([z^{-1}]),$$
$$\Phi_1(z) = \exp(\xi(t, zI))P^{(1)r}h^{-r},$$
$$\Psi_2^*(z) = \exp(-\xi(s, z^{-1}I))(h^{-l})^{\mathrm{T}}d([z^{-1}])(\tilde{P}^{(2)l})^{\mathrm{T}}.$$

Trivial computations give the following time evolution:

$$\partial_{t_n}P^{(1)l} = \left(L_1^n\right)_+P^{(1)l} - z^nP^{(1)l},$$
$$\partial_{t_n}\tilde{P}^{(2)r} = -d([z])\left(R_1^n\right)_-d([z^{-1}])\tilde{P}^{(2)r},$$
$$\partial_{t_n}P^{(1)r} = P^{(1)r}\left(h^{-r}\left(R_1^n\right)_-h^r + h^{-r}(\partial_{t_n}h^r) - z^nI\right),$$
$$\partial_{t_n}\tilde{P}^{(2)l} = \tilde{P}^{(2)l}\left(-h^{-l}\left(L_1^n\right)_+h^l + h^{-l}\left(\partial_{t_n}h^l\right)\right)$$

the last two can be simplified giving

$$\partial_{t_n}P^{(1)l} = \left(L_1^n\right)_+P^{(1)l} - z^nIP^{(1)l}, \tag{58}$$
$$\partial_{t_n}\tilde{P}^{(2)r} = -d([z])\left(R_1^n\right)_-d([z^{-1}])\tilde{P}^{(2)r}, \tag{59}$$
$$\partial_{t_n}P^{(1)r} = P^{(1)r}(h^{-r}\left(R_1^n\right)_{--}h^r - z^nI), \tag{60}$$
$$\partial_{t_n}\tilde{P}^{(2)l} = -\tilde{P}^{(2)l}\left(d([z^{-1}])h^{-l}\left(L_1^n\right)_{++}h^ld([z])\right), \tag{61}$$

where $\left(R_1^n\right)_{--}$ means the lower triangular part including the main diagonal and $\left(L_1^n\right)_{++}$ means the strictly upper diagonal part. This simplification can be obtained evaluating the terms $h^{-r}\left(\partial_{t_n}h^r\right)$ and $h^{-l}\left(\partial_{t_n}h^l\right))$ through Sato's equations. Also they can be obtained by observing that $P_N^{(1)r}$ and $P_N^{(2)l}$ are monic so that the derivative of the leading term is equal to 0. Then the proof is obtained as we did in the scalar case using forward and backward recursion relations (33), (35), (39) and (40). □

Corollary 6.5 (non-Abelian Ablowitz–Ladik semidiscrete zero-curvature equations). *The matrices $\mathcal{L}_n^r$ and $\mathcal{L}_n^l$ satisfy the following time evolution:*

$$\partial_{t_i/s_i}\mathcal{L}_n^l = \mathcal{M}^l_{t_i/s_i,n+1}\mathcal{L}_n^l - \mathcal{L}_n^l\mathcal{M}^l_{t_i/s_i,n}, \tag{62}$$
$$\partial_{t_i/s_i}\mathcal{L}_n^r = \mathcal{L}_n^r\mathcal{M}^r_{t_i/s_i,n+1} - \mathcal{M}^r_{t_i/s_i,n}\mathcal{L}_n^r. \tag{63}$$

Proof. These equations are nothing but compatibility conditions of recursion relations (47) and (48) with time evolution (56) and (57). □

Remark 6.6. It should be noted that, with respect to the equations originally written in [3], here we have two coupled non-Abelian Ablowitz–Ladik equations.

Example 6.7 (the first flows; non-Abelian analogue of discrete nonlinear Schrödinger). As we did for the scalar case we will compute the first matrices $\mathcal{M}^{r/l}_{t_1/s_1,k}$ and use them to construct the non-Abelian version of discrete nonlinear Schrödinger. We start with $\mathcal{M}^l_{t_1,k}$; (58) gives us immediately

$$\partial_{t_1} P^{(1)l}_k = P^{(1)l}_{k+1} - x^l_{k+1} y^r_k P^{(1)l}_k - z P^{(1)l}_k$$
$$= z P^{(1)l}_k + x^l_{k+1} \tilde{P}^{(2)r}_k - x^l_{k+1} y^r_k P^{(1)l}_k - z P^{(1)l}_k = -x^l_{k+1} y^r_k P^{(1)l}_k + x^l_{k+1} \tilde{P}^{(2)r}_k,$$

while we obtain immediately from (59) that

$$\partial_{t_1} \tilde{P}^{(2)r}_k = -z h^r_k h^{-r}_{k-1} \tilde{P}^{(2)r}_{k-1}.$$

Then we use recursion relation (39) combined with

$$h^r_k h^{-r}_{k-1} = \left(I - y^r_k x^l_k\right)$$

(this one comes from recursion relation (41) combined with t–s symmetry) to arrive to

$$\partial_{t_1} \tilde{P}^{(2)r}_k = z y^r_k P^{(1)l}_k - z P^{(2)r}_k.$$

These computations give us

$$\mathcal{M}^l_{t_1,k} = \begin{pmatrix} -x^l_{k+1} y^r_k & x^l_{k+1} \\ z y^r_k & -zI \end{pmatrix}. \tag{64}$$

Exploiting t–s symmetry we can immediately write

$$\mathcal{M}^l_{s_1,k} = \begin{pmatrix} z^{-1} I & -z^{-1} x^l_k \\ -y^r_{k+1} & y^r_{k+1} x^l_k \end{pmatrix}. \tag{65}$$

Analogue computations for $\mathcal{M}^r_{t_1}$ give us

$$\partial_{t_1} P^{(1)r}_k = P^{(1)r}_{k+1} - P^{(1)r}_k y^l_k x^r_{k+1} - z P^{1(r)}_k$$
$$= z P^{1(r)}_k + \tilde{P}^{(2)l}_k x^r_{k+1} - P^{(1)r}_k y^l_k x^r_{k+1} - z P^{1(r)}_k = \tilde{P}^{(2)l}_k x^r_{k+1} - P^{(1)r}_k y^l_k x^r_{k+1}$$

and

$$\partial_{t_1} \tilde{P}^{(2)l}_k = -z \tilde{P}^{(2)l}_{k-1} \left(h^{-l}_{k-1} h^l_k\right) = -\tilde{P}^{(2)l}_k z + P^{(1)r}_k z y^l_k$$

(here we started from (60) and (61) and we used recursion relations (37), (40), (42), the last one combined with t–s symmetry). Then we arrive at

$$\mathcal{M}^r_{t_1,k} = \begin{pmatrix} -y^l_k x^r_{k+1} & z y^l_k \\ x^r_{k+1} & -zI \end{pmatrix} \tag{66}$$

and using t–s symmetry again we also get

$$\mathcal{M}^r_{s_1,k} = \begin{pmatrix} z^{-1} I & -y^l_{k+1} \\ -z^{-1} x^r_k & x^r_k y^l_{k+1} \end{pmatrix}. \tag{67}$$

As we did for the scalar case we introduce times t_0 and s_0 that give matrices

$$\mathcal{M}^{r/l}_{t_0,k} = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \qquad \mathcal{M}^{r/l}_{s_0,k} = \begin{pmatrix} 0 & 0 \\ 0 & -I \end{pmatrix}. \tag{68}$$

Then we construct the matrices

$$\mathcal{M}^l_{\tau,k} = \mathcal{M}^l_{t_1,k} + \mathcal{M}^l_{s_1,k} - \mathcal{M}^l_{t_0,k} - \mathcal{M}^l_{s_0,k} = \begin{pmatrix} z^{-1} I - I - x^l_{k+1} y^r_k & x^l_{k+1} - z^{-1} x^l_k \\ z y^r_k - y^r_{k+1} & y^r_{k+1} x^l_k + I - zI \end{pmatrix}$$

$$\mathcal{M}^r_{\tau,k} = \mathcal{M}^r_{t_1,k} + \mathcal{M}^r_{s_1,k} - \mathcal{M}^r_{t_0,k} - \mathcal{M}^r_{s_0,k} = \begin{pmatrix} z^{-1} I - I - y^l_k x^r_{k+1} & z y^l_k - y^l_{k+1} \\ x^r_{k+1} - z^{-1} x^r_k & x^r_k y^l_{k+1} - zI + I \end{pmatrix}$$

associate with the time $\tau = t_1 + s_1 - t_0 - s_0$. Semidiscrete zero-curvature equations

$$\partial_\tau \mathcal{L}^l_k = \mathcal{M}^l_{\tau,k+1}\mathcal{L}^l_k - \mathcal{L}^l_k\mathcal{M}^l_{\tau,k}$$
$$\partial_\tau \mathcal{L}^r_k = \mathcal{L}^r_k\mathcal{M}^r_{\tau,k+1} - \mathcal{M}^r_{\tau,k}\mathcal{L}^r_k$$

are equivalent to the systems

$$\begin{cases}\partial_\tau x^l_k = x^l_{k+1} - 2x^l_k + x^l_{k-1} - x^l_{k+1}y^r_k x^l_k - x^l_k y^r_k x^l_{k-1}\\ \partial_\tau y^r_k = -y^r_{k+1} + 2y^r_k - y^r_{k-1} + y^r_{k+1}x^l_k y^r_k + y^r_k x^l_k y^r_{k-1}\end{cases} \tag{69}$$

$$\begin{cases}\partial_\tau x^r_k = x^r_{k+1} - 2x^r_k + x^r_{k-1} - x^r_{k-1}y^l_k x^r_k - x^r_k y^l_k x^r_{k+1}\\ \partial_\tau y^l_k = -y^l_{k+1} + 2y^l_k - y^l_{k-1} + y^l_{k-1}x^r_k y^l_k + y^l_k x^r_k y^l_{k+1}.\end{cases} \tag{70}$$

Note that both of them are equivalent to the discrete matrix nonlinear Schrödinger as written, for instance, in [6]. Using (69) and (70) together we perform the reduction to the Hermitian case in a different way from [6]. First of all we rescale $\tau \mapsto \mathrm{i}\tau$ and then we impose

$$y^r_k = \pm\left(x^r_k\right)^*$$
$$y^l_k = \pm\left(x^l_k\right)^*.$$

Note that this reduction (with the plus sign) corresponds to studying the theory of matrix orthogonal polynomials on the unit circle as described in [19, 22], hence it is very natural. This reduction gives us the two coupled equations

$$\begin{cases}-\mathrm{i}\partial_\tau x^l_k = x^l_{k+1} - 2x^l_k + x^l_{k-1} \mp x^l_{k+1}\left(x^r_k\right)^* x^l_k \mp x^l_k\left(x^r_k\right)^* x^l_{k-1}\\ -\mathrm{i}\partial_\tau x^r_k = x^r_{k+1} - 2x^r_k + x^r_{k-1} \mp x^r_{k-1}\left(x^l_k\right)^* x^r_k \mp x^r_k\left(x^l_k\right)^* x^r_{k+1}\end{cases} \tag{71}$$

already studied in [4] and generalized in [5].

Remark 6.8. In [16] the authors studied finite gap solutions of the Ablowitz–Ladik hierarchy. It could be interesting to generalize their results to the non-Abelian case. We will consider this problem in a subsequent publication.

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